REDUCING ONE REVERSE HEAT-CONDUCTION PROBLEM TO A SEMIREVERSE ONE

O. V. Minin

UDC 536.24.02

The algorithm for the solution of a reverse heat-conduction problem is simplified on the basis of earlier derived approximate solutions to the corresponding second boundary-value problem.

On the basis of the solution to the second boundary-value problem in heat conduction for a half-space, simple expressions have been derived in [1, 2] for calculating the temperature field $\vartheta(x, \tau)$ and the temperature gradient $\partial\vartheta(x, \tau)/\partial x$ within the hot zone of a body from temperature readings at two points.

The extrapolation formulas are:

$$\vartheta(x, \tau) = \vartheta_1 \left[\frac{x_2 - x}{x_2 - x_1} - \frac{x_1 - x}{x_2 - x_1}\left(\frac{\vartheta_2}{\vartheta_1}\right)^{1/n}\right]^n, \tag{1}$$

$$\frac{\partial\vartheta(x, \tau)}{\partial x} = -n\vartheta_1 \left[\frac{x_2 - x}{x_2 - x_1} - \frac{x_1 - x}{x_2 - x_1}\left(\frac{\vartheta_2}{\vartheta_1}\right)^{1/n}\right]^{n-1} \left[\frac{1}{x_2 - x_1} - \frac{1}{x_2 - x_1}\left(\frac{\vartheta_2}{\vartheta_1}\right)^{1/n}\right], \tag{2}$$

where

$$\vartheta_1 = \vartheta(x_1, \tau), \quad \vartheta_2 = \vartheta(x_2, \tau).$$

Here $n = 3$ when the thermal flux function $q(\tau) = \text{const}$, and $n = 7$ when $q(\tau) = b\tau$. It has been possible to establish that, within the same accuracy (3%) $n = 4$ when $q(\tau) = b\tau^{1/4}$, $n = 5$ when $q(\tau) = b\tau^{1/2}$, and $n = 6$ when $q(\tau) = b\tau^{3/4}$. Generally, $3 \le n \le 7$ when $q(\tau) = b\tau^{p/q}$, $p \le q$, $b = \text{const}$ and $b > 0$.

The error in determining the temperature and the temperature gradient in the hot zone of a body $0 \le x \le \sqrt{a\tau^*}$ does not exceed 3% for the entire class of thermal flux functions $q(\tau) = b\tau^{p/q}$ at a time

$$\tau^* \geqslant \frac{x_2^2}{a}, \tag{3}$$

with a denoting the thermal diffusivity of the material.

Let $x_2/x_1 = 2$. Then relations (1) and (2) for calculating the temperature and the thermal flux at the body surface ($x = 0$) become, respectively,

$$\vartheta(0, \tau) = \vartheta_1 \left[2 - \left(\frac{\vartheta_2}{\vartheta_1}\right)^{1/n}\right]^n, \tag{4}$$

$$q(\tau) = \lambda \frac{n}{x_1} \vartheta(0, \tau) \left[\frac{1 - \left(\frac{\vartheta_2}{\vartheta_1}\right)^{1/n}}{2 - \left(\frac{\vartheta_2}{\vartheta_1}\right)^{1/n}}\right]. \tag{5}$$

We will now consider the reverse heat-conduction problem which corresponds to the second boundary-value problem. As in the reverse problem in a half-space with boundary conditions of the first kind [3], the values of thermal flux at the body boundary can be determined from the following integral equations of the convolution kind:

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 24, No. 6, pp. 1138-1143, June, 1973. Original article submitted June 5, 1972.

© 1975 Plenum Publishing Corporation, 227 West 17th Street, New York, N.Y. 10011. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of the publisher. A copy of this article is available from the publisher for $15.00.

$$\vartheta(x, \tau) = \frac{\sqrt{a}}{\lambda\sqrt{\pi}} \int_0^{\tau} q(t) \frac{\exp\left[-\frac{x^2}{4a(\tau-t)}\right]}{\sqrt{\tau-t}}, \quad (6)$$

where $\vartheta(x, \tau)$ are the temperature readings at any point inside the body.

Problem (6), as has been shown in [4], is improper in the Adamar sense, inasmuch as the important requirement that solution $q(\tau)$ be a continuous function of the $\vartheta(x, \tau)$ readings is not necessarily satisfied. In other words, the solution becomes unstable with respect to "initial input data." An explanation for this is that the convolution operator in the reverse problem becomes, so to speak, improper [5]. The basic indication of an improper operator is that the inverse operator becomes unbounded.

Tikhonov has shown in [3, 4] that, in order to obtain a stable solution to such problems, it is necessary to apply the method of regularization.

The purpose of this article is to demonstrate that, with the aid of the extrapolation formulas (4)-(5) for solving Eq. (6), regularization is no longer required and the solution can be found by the conventional method of successive approximations.

Although the said class of thermal flux functions is limited in the mathematical sense, from the practical standpoint it is sufficiently important in that the parameter $b > 0$, which determines the rate of change of thermal flux, can vary over a wide range $0 \le b < +\infty$.

In the integral equation (6) we let $x = 0$ and then use Eqs. (4)-(5). Replacing τ by the Fourier number $\mathrm{Fo} = a\tau/x_1^2$ yields the equality

$$\vartheta_1\left[2-\left(\frac{\vartheta_2}{\vartheta_1}\right)^{1/n}\right]^n \simeq \frac{1}{\sqrt{\pi}} \int_0^{\mathrm{Fo}} \vartheta_1(\mathrm{Fo}')\left[2-\left(\frac{\vartheta_2}{\vartheta_1}\right)^{1/n}\right]^n n\left[\frac{1-\left(\frac{\vartheta_2}{\vartheta_1}\right)^{1/n}}{2-\left(\frac{\vartheta_2}{\vartheta_1}\right)^{1/n}}\right]\frac{d\mathrm{Fo}'}{\sqrt{\mathrm{Fo}-\mathrm{Fo}'}}, \quad (7)$$

with the dimensionless integration variable Fo' defined to the same scale as Fo.

Introducing a second point where the temperature is read, therefore, will convert the reverse problem for the given class of thermal functions $q(\tau)$ into a semireverse one.

Remembering that equality (7) is approximate, we may regard the power exponent n as a function of Fo. Evidently, at large values of τ the equation becomes more exact and n becomes less dependent on τ.

We note further that at any time $\tau \ge 0$ the function

$$\psi(\mathrm{Fo}', n) = n\left[\frac{1-\left(\frac{\vartheta_2}{\vartheta_1}\right)^{1/n}}{2-\left(\frac{\vartheta_2}{\vartheta_1}\right)^{1/n}}\right] \quad (8)$$

has a limited range of variation with $3 \le n \le 7$, its maximum being $\max_{n,\mathrm{Fo}} \psi = 7/2$. Adding ψ to the kernel of Eq. (7), which now can be represented as

$$\tilde{K}_n = \begin{cases} \frac{\psi(\mathrm{Fo}', n)}{\sqrt{\mathrm{Fo}-\mathrm{Fo}'}}, & 0 \leqslant \mathrm{Fo}' < \mathrm{Fo}, \\ 0 & \mathrm{Fo}' \geqslant \mathrm{Fo}, \end{cases} \quad (9)$$

with $\mu = 1/\sqrt{\pi}$ and $^{1/n}]^n = \varphi(n, \mathrm{Fo})$, we can write

$$\varphi(n, \mathrm{Fo}) \simeq \mu \int_0^{\mathrm{Fo}} \tilde{K}_n(\mathrm{Fo}, \mathrm{Fo}', n)\, \varphi(n, \mathrm{Fo}')\, d\mathrm{Fo}' \quad (10)$$

or

$$\varphi(n, \mathrm{Fo}) - \mu\tilde{K}_n\varphi \simeq 0, \quad (11)$$

where

$$\tilde{K}_n\varphi = \int_0^{\mathrm{Fo}} \tilde{K}_n(n, \mathrm{Fo}, \mathrm{Fo}')\, \varphi(n, \mathrm{Fo}')\, d\mathrm{Fo}'. \quad (12)$$

Structurally, Eq. (11) resembles the homogeneous Volterra equation of the second kind. This equation is already not improper in the earlier sense, since we have now an equation of the second kind rather than of the first kind.

It must now be proved that the method of successive approximations (with respect to n) is convergent and that such an iteration process is stable.

We will regard the value of exponent n as the parameter of Eq. (11). This immediately poses the question as to the character of operator $\tilde{K}_n\varphi$ containing that numerical parameter.

It is well known [6] that, for obtaining unique solutions to Fredholm and Volterra equations of the second kind, the method of successive approximations is convergent when the integral operator Ky is compressible. This means that for functions y_m and y_{m+1} defined on two adjacent intervals m and m + 1 of the iteration path we have the inequality

$$\|Ky_m - Ky_{m+1}\| \leqslant L\,\|y_m - y_{m+1}\|, \tag{13}$$

with the Lipschitz constant L < 1 and the norm sign $\|\cdot\|$ for the corresponding functional space.

We will verify that the operator $\tilde{K}_n\varphi$ in Eq. (11) is compressible, under definite accuracy requirements, i.e., that condition (13) is satisfied with L < 1.

By virtue of the operator $\tilde{K}_n\varphi$ being positive and monotonic with respect to n and Fo for every $n_m \in [3, 7]$, the following estimate is valid:

$$\|\tilde{K}_n\varphi(n_m, \mathrm{Fo}) - \tilde{K}_n\varphi(n_{m+1}, \mathrm{Fo})\| = \|\tilde{K}_n^{(m)}\varphi(n_m, \mathrm{Fo}) - \tilde{K}_n^{(m+1)}\varphi(n_{m+1}, \mathrm{Fo})\|$$
$$< \|\tilde{K}_n^{(m+1)}(\varphi(n_m, \mathrm{Fo}) - \varphi(n_{m+1}, \mathrm{Fo}))\| < \max_{n,\mathrm{Fo}} \int_0^{\mathrm{Fo}} \tilde{K}_n d\,\mathrm{Fo}'\, \|\varphi(n_m, \mathrm{Fo}) - \varphi(n_{m+1}, \mathrm{Fo})\|, \tag{14}$$

with the expression $\max\limits_{\mathrm{Fo},n} |\varphi(n_m, \mathrm{Fo}) - \varphi(n_{m+1}, \mathrm{Fo})|$ regarded as the norm. The cofactor in inequality (14) is

$$\max_{n,\mathrm{Fo}'} \int_0^{\mathrm{Fo}} \tilde{K}_n d\mathrm{Fo}' = \frac{14}{\sqrt{\pi}}\,\varepsilon\sqrt{\mathrm{Fo}} < 1 \tag{15}$$

if

$$\varepsilon = \max_{n,\mathrm{Fo}'} \frac{1}{n}\,\psi(\mathrm{Fo}', n) = \frac{1 - \left(\dfrac{\vartheta_2}{\vartheta_1}\right)^{1/3}}{2 - \left(\dfrac{\vartheta_2}{\vartheta_1}\right)^{1/3}} < \sqrt{\frac{\pi}{256\,\mathrm{Fo}}}, \tag{16}$$

and from here we find that the inequality

$$\left(1 - \frac{\sqrt{\pi}}{\sqrt{\dfrac{256\,\mathrm{Fo}}{\pi}} - 1}\right)^3 < \frac{\vartheta_2}{\vartheta_1} < 1. \tag{17}$$

must hold true.

We have established an important relation between the temperature readings and the time parameters. Relation (17) yields an overestimate of the lower convergence limit, with regard to τ, for the method of successive approximations as applied to the calculation of n. On the other hand, according to estimate (3), the error in the determination of temperatures and temperature gradients at every point of the region

$$0 \leqslant x \leqslant \sqrt{a\tau^*}$$

does not exceed 3%, if

$$\tau^* \geqslant \frac{x_2^2}{a}.$$

From here we obtain

$$\mathrm{Fo}_c^* = \frac{\tau^* a}{x_2^2} = \frac{\tau^* a}{4x_1^2} = \frac{1}{4}\,\mathrm{Fo}^* \geqslant 1. \tag{18}$$

Inserting $Fo^* = 4$ into inequality (17) will yield

$$0.67 \leqslant \vartheta_2/\vartheta_1 < 1. \qquad (19)$$

This inequality replaces condition (3) by a more convenient one, inasmuch as the thermal diffusivity does not appear in (19).

If $\vartheta_2/\vartheta_1 \geq 0.67$, therefore, then the method of successive approximations will converge almost uniformly with respect to time and the series of thus found n_m values will have a fixed point $\bar{n}$ which is the solution to the equation. In other words, an indicator of uniform convergence $n_m \to \bar{n}$ is the stability of n_m values after time $\tau_0 \geq \tau^*$. Since $\tilde{K}_n\varphi$ is a Volterra operator [6], hence the sequence of n_m values will converge to $\bar{n}$ at least as fast as a geometric progression. The error in the determination of $\bar{n}$ will depend here, first of all, on how accurately the true thermal flux at the boundary can be approximated by a function of $q(\tau) = b\tau^{p/q}$ class. Errors in the calculation of $\bar{n}$ will also depend on the selected number of nodes in the interpolation polynomials $\vartheta_1(t)$ and $\vartheta_2(t)$, and on the rounding-off errors in the final count. The initial approximation n_0 can be picked arbitrarily within the range $3 \leq n \leq 7$.

It can be easily verified that calculating $n_m \to \bar{n}$ is a stable process, i.e., that $\Delta n_m/n_m \to 0$ when the number of steps m increases. We will not show here the proof of stability, however, because its content and terminology are beyond the scope of this article. Problems concerning the stability of the method of successive approximations when applied to equations of the $Ax = f$ kind (A denoting an operator with a bounded inverse operator $\|A^{-1}\|_{E_1}$) have been analyzed theoretically in [5, 6].

If we lower the requirements as to accuracy of the approximations to the exact solution, then constraint (3) will weaken appreciably, and with it also constraint (19). Thus, with a 10% error allowed in the determination of the temperature within the hot zone, inequality (19) will become

$$0.38 \leqslant \frac{\vartheta_2}{\vartheta_1} < 1.$$

We will now show the scheme for solving Eq. (7), which has been written with a rather simple notation in the ALGOL-60 language.

From the continuous recording of the temperature $\vartheta_1(x_1, \tau)$ and $\vartheta_2(x_2, \tau)$ as functions of time, we tabulate the temperature readings $\vartheta(x_1, \tau_i)$ and $\vartheta(x_2, \tau_i)$ $(i = 1, 2, \ldots, N)$ at uniform intervals τ. For each given time τ_i we determine the upper interpolation limit in Eq. (7), and from the sequence of nodes $\vartheta(x_1, \tau_{i-r})$ and $\vartheta(x_2, \tau_{i-r})$ $(r = 1, 2, 3, \ldots i)$ we construct Newtonian interpolation polynomials depending on $t < \tau_i$. This is necessary in order to perform the interpolation. Practical estimates indicate that the interpolation polynomials $\vartheta_1(\tau_{i-r})$ and $\vartheta_2(\tau_{i-r})$ must be of the fifth degree at least.

We start with an arbitrary initial approximation $n_0 \in [3, 7]$ and calculate the right-hand side of the equation. Since $\vartheta(x_1, \tau_i)$ and $\vartheta(x_2, \tau_i)$ on the left-hand side of the equation are essentially numbers, hence we proceed to the next approximation $n = n_1$. Then we insert again, but this time n_1, into the right-hand side and find n_2. This process is repeated until $(n_{m+1} - n_m)/n_{m+1} \leq \alpha$, α denoting the stipulated accuracy for n.

For every τ_i, as has been mentioned earlier, the exponent n_m is some function of τ. Consequently, having ascertained that $\vartheta_2/\vartheta_1 \geq 0.67$ (or $\vartheta_2/\vartheta_1 \geq 0.38$), we may assume that $n_m = \bar{n} = \text{const}$ and let the computer print out its value.

The thus found value of $\bar{n}$ determines the magnitude of the thermal flux $q(\tau)$ as a function of time, at time $\tau_i > \tau^*$.

From a series of temperature readings at two points of a body it is thus possible to diagnose the outside thermal effect and to determine the temperature field as well as the temperature gradient, without knowing the thermophysical properties of the material.

If the thermal flux at the boundary varies as a function of time $q(\tau)$ not in the class considered here, then it becomes necessary to regularize by the Tikhonov or Morozov method, as has been done, for instance, by Alifanov [7, 8].

LITERATURE CITED

1. O. V. Minin and N. A. Yaryshev, Inzh.-Fiz. Zh., 17, No. 3 (1969).
2. O. V. Minin and N. A. Yaryshev, Problemy Prochnosti, No. 3 (1972).

3. A. N. Tikhonov and V. B. Glasko, Zh. Vyschislit. Matem. i Matem. Fiz., 7, No. 4 (1967).
4. A. N. Tikhonov, Dokl. Akad. Nauk SSSR, 151, No. 3 (1963); 153, No. 1 (1963).
5. M. K. Gavurin, Lectures on Computation Methods [in Russian], Nauka, Moscow (1971).
6. S. G. Krein (editor), "Functional Analysis," in: Mathematical Textbook Library [in Russian], Nauka, Moscow (1972).
7. O. M. Alifanov, Inzh.-Fiz. Zh., 23, No. 6 (1972).
8. O. M. Alifanov, ibid., 24, No. 2 (1973).